Classification of Du Val Singularities

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Motivations and Background

Mathematicians enjoy resolving singularities, because many nice properties and theorems only applies to nonsingular things. So comes the desire to classify singularities whenever we can.

Resolution of a singularity $x \in X$ means finding a proper birational map $f : Y \to X$ s.t. Y is nonsingular and $f : Y \backslash f^{-1}(x) \to X \backslash x$ is an isomorphism. So x is mapped by curves (Riemann surfaces) on the nonsingular surface Y.

Theorem (Abhyankar...)

The singularities of any surface can be resolved.

Motivations and Background

Theorem (Resolution of imbedded curve singularities, SH IV(4.1.1))

For any irreducible curve $C \subset X$ a nonsingular surface, there exists another surface Y and a regular map $f : Y \to X$ s.t. f is a composite of blow-ups and the birational transform of C is nonsingular on Y.

Theorem (HS V(5.5))

Let $T : X \to X'$ be a birational transform of surfaces. Then T can be factored into a finite sequence of monoidal tranformations (blow-ups at a point) and their inverses (blow-downs at a point).

Therefore we can classify a singularity by classifying the sequence of exceptional curves resulted from the blow-ups. To do so, we analyze them using the idea of "intersection numbers." And we can define them s.t. they agree with our intuition of intersections of curves (Riemann surfaces).

On the other hand, we should only be concerned with the minimal resolution, which is a resolution where the blow-up steps do not produce unnecessary (contractible) exceptional curves.

Theorem (from Castelnuovo's criterion)

All (-1)-curves can be contracted.

Here (-1)-curves refers to curves that are isomorphic to $\mathbb{P}^1$ with self intersection number -1.

Motivations and Background

The canonical classes of a surface holds important status, so we would like to leave them alone when resolving singularities

Du Val Singularities

A point $x \in X$ of a normal surface is called *a Du Val singularity* if there exists a minimal resolution $f : Y \to X$ contracting curves $C_1, \ldots, C_r$ to x s.t. $K_Y C_i = 0$ for all i, where K_Y is the canonical class of Y.

A Catch-22: classifying Du Val singularities leads us to the magical Dynkin Diagrams, so we would like to classify Du Val singularities.

More Background

Theorem (Contracted curves of a point, SH IV(4.2.2))

Let $f : Y \to X$ be a resolution of the singularity x on a surface X, where the inverse image of x is $C_1 \cup \cdots \cup C_r$. Then the matrix $\{C_i C_j\}$ is negative definite.

Theorem (Adjunction Formula)

For any curve $C \subset X$, the canonical class K_X of the surface and canonical class K_C of the curve satisfies

$$deg\ K_C = C(C + K_X)$$

Theorem (Degree Genus Formula)

For any nonsingular curve $\overline{C}$ with genus $g(\overline{C}) = g$, its canonical class $K_{\overline{C}}$ satisfies

$$deg\ K_{\overline{C}} = 2g - 2$$

More Background

Properties of Blowing Up a surface at a point

If we have the blow up X' of a surface X at a smooth point $x \in C \subset X$ where C is a curve, then $\sigma : X' \to X$ induces $\sigma' : C \to C'$ the birational (strict) transform of C and $\sigma^* : C \to C^*$ the total transform of C. Below we let L be the exceptional curve of the blow up, and we write the multiplicity of a point $x \in C \subset X$ as $\mu_x(C) = k$.

1. $\sigma^*(C) = \sigma'(C) + kL$
2. $K_{X'} = \sigma'(K_X) + L$
3. $\sigma^*(D)L = 0 \quad \forall D \subset X$
4. $\sigma^*(D_1) \cdot \sigma^*(D_2) = D_1 D_2 \quad \forall D_i \subset X$
5. $L \sim \mathbb{P}^1$ and $L^2 = -1$

Conclusions

Thus we can conclude

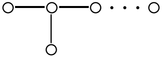
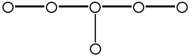
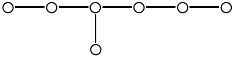
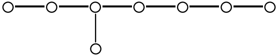
$$C_i^2 = -2 \tag{1}$$

$$C_i C_j = 0 \text{ or } 1 \tag{2}$$

Since classifying Du Val singularities is equivalent to classifying $\{C_i\}_{i=1}^r$, the above relation shows that it is also equivalent to classifying the negative definite lattice $\mathbb{Z}e_1 + \ldots + \mathbb{Z}e_r$ where $e_i^2 = -2, e_i e_j > 0$, since we have $C_i \sim e_i$.

Therefore Dynkin Diagrams

Results

Name	Equation	Group	Resolution Graph
A_n	$x^2 + y^2 + z^{n+1}$	cyclic $(n+1)$	
D_n	$x^2 + y^2 z + z^{n-1}$	binary dihedral $(n-2)$	
E_6	$x^2 + y^3 + z^4$	binary tetrahedral	
E_7	$x^2 + y^3 + yz^3$	binary octahedral	
E_8	$x^2 + y^3 + z^5$	binary icosahedral	

Dynkin Diagrams vs. Du Val singularities

References

Igor R. Shafarevich (1972)
Basic Algebraic Geometry I

Robin Hartshorne (1977)
Algebraic Geometry

And most importantly my graduate student: Zhu, Yuecheng!

And drumrolls...

Now we can classify the Du Val singularities through some simple algebraic manipulations. First we have the following regarding resolution of embedded curve singularities.

$$\begin{aligned}
\deg K_{\sigma'(C)} &= \sigma'(C)(\sigma'(C) + K_{X'}) \\
&= (\sigma^*(C) - kL)(\sigma^*(C) - kL + \sigma^*(K_X) + L) \\
&= \sigma^*(C)\sigma^*(C) - k\sigma^*(C)L + \sigma^*(C)\sigma^*(K_X) + \dots \\
\sigma^*(C)L &- k\sigma^*(C)L - k\sigma^*(K_X)L + k(k-1)L^2 \\
&= C^2 - 0 + CK_X + 0 - 0 - 0 - k(k-1) \\
&= C(C + K_X) - k(k-1)
\end{aligned}$$

Still drumrolls...

Then applying the above to full sequence of resolution of a Du Val singularity gets us

$$\begin{aligned}
\deg K_{\overline{C}} &= \overline{C}(\overline{C} + K_{\overline{X}}) \\
2g(\overline{C}) - 2 &= C(C + K_X) - \sum_i k_i(k_i - 1) \\
\Rightarrow \quad 2 + C(C + K_X) &= 2g(\overline{C}) + \sum_i k_i(k_i - 1) \geq 0 \\
\Rightarrow \quad C(C + K_X) &\geq 2 \\
\Rightarrow \quad C_i(C_i + K_Y) &= C_i^2 + 0 \geq 2 \\
\Rightarrow \quad C_i^2 &= -2
\end{aligned}$$

Drum guy's hands are getting tired...

Last but not least, let's apply the theorem about contracted curves to our case. Let $\alpha = (0, \ldots, 1, \ldots, 1, \ldots, 0)^T$, the vector with identity on i, j location and 0 otherwise. Therefore

$$
\begin{aligned}
& \{C_i C_j\} \text{ is negative definite} \\
\Rightarrow \quad & b(\alpha, \alpha) = \alpha^T (C_i C_j) \alpha < 0 \\
\Rightarrow \quad & (C_i + C_j)^2 < 0 \\
& C_i^2 + C_j^2 + 2 C_i C_j < 0 \\
& C_i C_j < 2 \\
\Rightarrow \quad & C_i C_j = 0 \text{ or } 1
\end{aligned}
$$